The He II Theory Preserving the Symmetry of the Initial Hamiltonian of the System

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(Dated: February 5, 2013)

The article suggests a method for the construction of the He II theory preserving the symmetry of the initial Hamiltonian.

PACS numbers: 67.85.Hj, 67.85.Jk

I. INTRODUCTION

In 1947 Bogoliubov suggested a microscopic description of He II [1]. The main idea of this description is the selection of the two-particle interaction terms in the Hamiltonian $\sim a_0^+ a_0$, $a_0^+ a_0^+$, $a_0 a_0$, where $a_{\mathbf{p}}^+$ and $a_{\mathbf{p}}$ are the creation and annihilation operators for He^4 atoms with momentum $\mathbf{p}$. Accounting for these terms only in order to calculate the physical parameters of the system is not in doubt, because the condensation of a macroscopic number of atoms in a state with zero momentum results in the fact that the transition matrix elements corresponding to the selected terms are far superior in magnitude to the rest of the matrix elements.

However, for the diagonalization procedure Bogolyubov replaced operators a_0^+ and a_0 with so-called c-numbers. As a result, the truncated Hamiltonian, in contrast to the initial one, no longer keeps the number of particles in the system, which means the breach of the initial Hamiltonian symmetry (IHS) of the system. This state of affairs will always be a concern in the scientific community [2].

This paper considers diagonalization procedure [3], which can be the basis for the He II theory preserving the symmetry of the initial Hamiltonian.

II. DIAGONALIZATION PROCEDURE

So, let us consider the truncated Hamiltonian

$$\hat{H}_{trunc} = \sum_{\mathbf{q}} \hat{h}_{\mathbf{q}} \tag{1}$$

with

$$\hat{h}_{\mathbf{q}} = t_{\mathbf{q}} a_{\mathbf{q}}^+ a_{\mathbf{q}} + \frac{V_{\mathbf{q}}}{2\Omega} a_{\mathbf{q}}^+ a_{-\mathbf{q}}^+ a_0 a_0 + \frac{V_{\mathbf{q}}}{2\Omega} a_0^+ a_0^+ a_{\mathbf{q}} a_{-\mathbf{q}} + \frac{V_{\mathbf{q}}}{\Omega} a_0^+ a_{\mathbf{q}}^+ a_{\mathbf{q}} a_0, \tag{2}$$

where Ω is the system volume. Hereinafter we will not consider the terms $\sim V_0$, because they only add a constant to the energy of the system with a fixed number of particles.

Let us find the ground state of the system Φ_0 as

$$\Phi_0 = \frac{a_0^{+N}}{\sqrt{N!}} |0\rangle + \frac{1}{2} \sum_{\mathbf{q}} \sum_{0<m\le N/2} \alpha_m^{\mathbf{q}} \frac{a_{\mathbf{q}}^{+m} a_{-\mathbf{q}}^{+m}}{m!} \frac{a_0^{+(N-2m)}}{\sqrt{(N-2m)!}} |0\rangle, \tag{3}$$

where N is the number of particles in the system and

$$\alpha_m^{\mathbf{q}} = \alpha_m^{-\mathbf{q}}. \tag{4}$$

Then the diagonalization of $\hat{H}_{trunc}$ within space of states linked with right hand side of (3)

$$\Lambda = \left\{ \frac{a_{\mathbf{q}}^{+m} a_{-\mathbf{q}}^{+m}}{m!} \frac{a_0^{+(N-2m)}}{\sqrt{(N-2m)!}} |0\rangle : 0 \le m \le N/2, q > 0 \right\} \tag{5}$$

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is decoupled into independent diagonalizations of Hamiltonians $\hat{h}_{\mathbf{q}}$ in the subspaces

$$\lambda_{\mathbf{q}} = \left\{ \frac{a_{\mathbf{q}}^{+m} a_{-\mathbf{q}}^{+m}}{m!} \frac{a_0^{+(N-2m)}}{\sqrt{(N-2m)!}} |0\rangle : 0 \leq m \leq N/2 \right\}, \tag{6}$$

which gives equations for the eigenvalues $\varepsilon_{\mathbf{q}}$:

$$\begin{array}{l} \frac{V_{\mathbf{q}}}{\Omega} m \sqrt{(N-2m+2)\,(N-2m+1)} \alpha_{m-1}^{\mathbf{q}} \\ +2mt_{\mathbf{q}} \alpha_m^{\mathbf{q}} + 2\frac{V_{\mathbf{q}}}{\Omega} m\,(N-2m)\,\alpha_m^{\mathbf{q}} \\ +\frac{V_{\mathbf{q}}}{\Omega}\,(m+1)\,\sqrt{(N-2m-1)\,(N-2m)} \alpha_{m+1}^{\mathbf{q}} = \varepsilon_{\mathbf{q}} \alpha_m^{\mathbf{q}}. \end{array} \tag{7}$$

An additional definition will be useful in (7):

$$\alpha_{-1}^{\mathbf{q}} = 0, \alpha_0^{\mathbf{q}} = 1 \tag{8}$$

for its convenient connection with (3) at $m = 0$.

Let us now introduce operators $\varphi_{\mathbf{q}}^{+}$ and $\varphi_{\mathbf{q}}$:

$$\begin{array}{l} \varphi_{\mathbf{q}}^{+} = \chi_{\mathbf{q}}^{0} a_{\mathbf{q}}^{+} a_0 + \chi_{\mathbf{q}}^{-\mathbf{q}} a_0^{+} a_{-\mathbf{q}}, \\ \varphi_{\mathbf{q}} = \chi_{\mathbf{q}}^{0} a_0^{+} a_{\mathbf{q}} + \chi_{\mathbf{q}}^{-\mathbf{q}} a_{-\mathbf{q}}^{+} a_0, \end{array} \tag{9}$$

where

$$\chi_{\mathbf{q}}^{\mathbf{p}} = \frac{t_{\mathbf{p}+\mathbf{q}} - t_{\mathbf{p}} + \omega_{\mathbf{q}}}{2\sqrt{N t_{\mathbf{q}} \omega_{\mathbf{q}}}}. \tag{10}$$

Such a definition of $\varphi_{\mathbf{q}}^{+}$ and $\varphi_{\mathbf{q}}$ corresponds to the implementation of the equation

$$\left[\hat{H}, \varphi_{\mathbf{q}}^{+}\right] = \omega_{\mathbf{q}} \varphi_{\mathbf{q}}^{+} \tag{11}$$

in random phase approximation in the case when all the atoms are condensed at the low energy level, where

$$\hat{H} = \sum_{\mathbf{q}} t_{\mathbf{q}} a_{\mathbf{q}}^{+} a_{\mathbf{q}} + \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \frac{V_{\mathbf{q}}}{2\Omega} a_{\mathbf{p}+\mathbf{q}}^{+} a_{\mathbf{k}-\mathbf{q}}^{+} a_{\mathbf{k}} a_{\mathbf{p}} \tag{12}$$

is the total Hamiltonian of the system, and

$$\begin{array}{l} \omega_q = \sqrt{t_q\,(t_q + 2\Delta_q)}, \\ \Delta_q = V_q N/\Omega. \end{array} \tag{13}$$

Obviously, in view of (11) the lowest energy $\varepsilon_{\mathbf{q}}$ is reached when

$$\varphi_{\mathbf{q}} \Phi_0 = 0. \tag{14}$$

Here the action of $\varphi_{\mathbf{q}}$ should be limited to the subspace $\lambda_{\mathbf{q}}$. Thus, passing to the limit $N \to \infty$ in case of $m \geq 0$ we have

$$\alpha_m^{\mathbf{q}} = \left(\frac{t_{\mathbf{q}} - \omega_{\mathbf{q}}}{t_{\mathbf{q}} + \omega_{\mathbf{q}}}\right)^m. \tag{15}$$

On the other hand, in the same limit, it follows from (7) at $m = 0$ that

$$\Delta_{\mathbf{q}} \alpha_1^{\mathbf{q}} = \varepsilon_{\mathbf{q}} \alpha_0^{\mathbf{q}} \tag{16}$$

The combined consideration of (15, 16) gives

$$\varepsilon_{\mathbf{q}} = \omega_{\mathbf{q}} - t_{\mathbf{q}} - \Delta_{\mathbf{q}}, \tag{17}$$

and with this choice of $\varepsilon_{\mathbf{q}}$, Eq. (7) is also satisfied at $m > 0$.

The action of operators $h_{\mathbf{q}}^{+n_{\mathbf{q}}} h_{-\mathbf{q}}^{+n_{-\mathbf{q}}} / \sqrt{n_{\mathbf{q}}! n_{-\mathbf{q}}!}$ on Φ_0 within subspace

$$\tilde{\lambda}_{\mathbf{q}} = \left\{ \frac{a_{\mathbf{q}}^{+m} a_{-\mathbf{q}}^{+n}}{\sqrt{m!n!}} \frac{a_0^{+(N-m-n)}}{\sqrt{(N-m-n)!}} |0\rangle : m \geq 0, n \geq 0, m+n \leq N \right\} \tag{18}$$

gives the excited states of the system. Thus, the system energy E takes the form

$$E = \frac{1}{2} \sum_{\mathbf{q}} \varepsilon_{\mathbf{q}} + \sum_{\mathbf{q}} n_{\mathbf{q}} \omega_{\mathbf{q}}. \tag{19}$$

This is completely consistent with Bogoliubov's results for the excitation spectrum of the system.

III. DISCUSSION.

However, it may turn out that Bogoliubov's approach is not equivalent to the approach suggested in this paper. As a matter of fact, in contrast to Bogoliubov, I did not break IHS of the system during the diagonalization of its Hamiltonian. Therefore, the final answer to the question of the equivalence of the two approaches can be given only after a careful comparison of the results of calculations of the system physical quantities, for which the symmetry breach in the Bogoliubov theory is sensitive, for example, the superfluid flow in the Josephson effect [4]. Common to all the effects of this kind will be the presence of products with the structure uv in the equations for the physical quantities, where u and v are the parameters of the Bogoliubov transformation. If the suggested approach gives no terms corresponding to expressions $\sim uv$ in the Bogoliubov theory, then presumably the scientific community will be faced with the choice of the correct He II theory. In any case, this kind of research will help to clarify the origins of the paradox [5] discussed since 1964 associated with the interpretation of the Josephson effect in the BCS theory [6].

Acknowledgments

The author is grateful to Professor A.A. Rukhadze for the moral support of author's works on theories of superphenomena. In addition, the author thanks D.Sc. S.A. Trigger for criticism of author's first attempts to solve the problem, which were subsequently refused

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